ACADEMIC PRESS

Available online at www.sciencedirect.com

SCIENCE DIRECT

J. Differential Equations 188 (2003) 1–32

Journal of Differential Equations

http://www.elsevier.com/locate/jde

Existence and stability of solitary-wave solutions of equations of Benjamin–Bona–Mahony type

Lei Zeng

Department of Mathematics, Alabama Agriculture and Mechanical University, P.O. Box 326, Normal, AL 35762, USA

Received January 8, 2001; revised January 25, 2002

Abstract

We consider solitary-wave solutions of equations of Benjamin–Bona–Mahony type. We show that for a large class of equations of BBM type, there do exist stable sets consisting of solitary-wave profile functions. In the case of generalized BBM equations, we found that there are profile functions of stable solitary waves that are not the minimizers of the associated variational problem. Such a phenomenon is not known to exist for equations of Korteweg–de Vries type.

1. Introduction

This paper is concerned with the existence and stability of solitary-wave solutions of the equations of the form

$$u_t + f(u)_x + Mu_t = 0, \tag{1.1}$$

where $u = u(x, t)$ and f are real-valued functions, and M is a Fourier transform operator defined by

$$\widehat{Mu}(k) = m(k)\hat{u}(k),$$

where circumflexes denote Fourier transform and $m(k)$ is a even and real-valued function. The condition on $m(k)$ assures that the operator M takes real-valued functions to real-valued functions.

E-mail address: lzeng@aamu.edu.

0022-0396/03/$ - see front matter
PII: S0022-0396(02)00061-X

Eq. (1.1) describes mathematically the unidirectional propagation of non-linear dispersive waves. A prototypical example of an equation of type (1.1) is the well-known Benjamin–Bona–Mahony (BBM) equation

$$u_t + u_x + uu_x - u_{xxt} = 0, \tag{1.2}$$

which occurs when $f(u) = u + \frac{u^2}{2}$ and $m(k) = k^2$. Eq. (1.2) was proposed in [7] as a alternative to the Korteweg–de Vries (KDV) equation [16]

$$u_t + u_x + uu_x + u_{xxx} = 0 \tag{1.3}$$

for modeling water waves of small amplitude and large wavelength. In all these equations, u denotes a wave amplitude or velocity, x is proportional to the physical distance and t is proportional to the elapsed time.

If the non-linear terms of Eqs. (1.2) and (1.3) are replaced by $u^p u_x$ for $p > 0$, the resulting equations (called the generalized KdV and generalized BBM equations) read

$$u_t + u_x + u^p u_x + u_{xxx} = 0 \tag{1.4}$$

and

$$u_t + u_x + u^p u_x - u_{xxt} = 0. \tag{1.5}$$

A solitary-wave solution of a wave equation such as (1.1) is a traveling wave solution of the form $u(x,t) = \phi_c(x - ct)$ where ϕ_c is a localized wave profile function, which in general depends on the wavespeed c. (Usually the condition that ϕ_c be localized is interpreted to mean at least that $\phi_c(x) \to 0$ as $|x| \to \infty$.) Such a solitary-wave solution is said to be stable if for every $\varepsilon > 0$, there exists a $\delta > 0$ such that if

$$||u_0 - \phi_c|| < \delta$$

then the solution of (1.1) with $u(\cdot, 0) = u_0$ satisfies

$$\inf_{y \in \mathbb{R}} \to ||u(\cdot, t) - \phi_c(\cdot + y)|| < \varepsilon$$

for all $t \in \mathbb{R}$. (Here the norm is that of a Banach space in which the initial-value problem for the equation is well-posed.)

The first rigorous proof of the stability of solitary waves for an equation like (1.1) was given for the KdV equation by Benjamin [6], and Benjamin's proof was subsequently improved by Bona to allow less restrictive hypotheses [8]. Later it was proved by Weinstein [22] that the generalized KdV equation has stable solitary waves for all $p < 4$.

A more general class of equations of KdV type of the form

$$u_t + f(u)_x - Mu_x = 0 \tag{1.6}$$

was investigated by Bona et al. [9]. They showed that, if the solitary-wave solutions exist for wavespeeds ranging over an interval and a certain linear operator associated with the solitary wave has one negative simple eigenvalue and a simple zero eigenvalue, then whether or not a solitary wave is stable is determined by the convexity of a certain function of the solitary wave speed. When applied to (1.4), their results show that all solitary-wave solutions of (1.4) are stable if $p<4$ and all are unstable if $p>4$. (The case $p=4$ is still open; cf. [23]) The stability theory of [9] for (1.6) has been extended to equations of type (1.1) by Souganidis and Strauss [21]. In particular, in [21] it is shown that for the generalized BBM equation (1.5), all solitary waves are stable when $p \leqslant 4$, and when $p>4$, there is a critical value of solitary wave speed $c_r>1$, such that the solitary wave is stable for wave speed $c>c_r$ and unstable for $1<c \leqslant c_r$.

In circumstances when the assumptions of the theory in [9,21] can be verified, the results of these papers give sharp conditions for determining the stability or instability of solitary-wave solutions of Eqs. (1.1) and (1.6). However, the verification of these assumptions does not seem to be easily accomplished for general classes of symbols $m(k)$ of the Fourier multiplier operator M; nor is it easy in general to check whether the condition for stability holds for a given solitary wave.

Lions developed a general method to solve a class of variational problems which do not satisfy the compactness conditions required for classical methods of solution [17,18]. The centerpiece of this method is the concentration compactness lemma, which states that every sequence of positive L^1 functions whose L^1 norms are held constant has a subsequence with one of the three properties: vanishing, dichotomy or compactness (cf. Lemma 2.6). Lions and Cazenave observed in [10] that the method could be used to prove existence and stability of solitary waves for the non-linear Schrödinger equation. The method has since been adapted by different authors to handle a variety of model equations for water waves [1,11,13,14].

The typical setting for applying this method involves a constrained variational problem whose functional to be minimized and constraint functional are invariants of motion of the equation in question. The Euler–Lagrange equation of the variational problem is the equation to be satisfied by the solitary wave profile functions. The concentration compactness lemma is used to determine if the set of minimizers exists. If so, it is a set which consists of solitary wave profile functions and which is stable in the sense that if the initial data is close to the set, then the solution to the initial-value problem will remain close to it for all time. This notion of stability is in general broader (possibly weaker) than that mentioned above in that it asserts the stability of a set consisting of possibly different solitary-wave profile functions rather than the stability of the set of translates of a individual solitary-wave solution. If it is known that the set of minimizers consists of only translates of discrete solitary-wave profile functions, then the two notions of stability coincide.

Albert [1] and Albert and Linares [3] used the concentration compactness method to study the solitary-wave solutions of Eq. (1.6) and obtained existence and stability results for a general class of functions $m(k)$. We apply the method to Eq. (1.1) and obtain similar results, which can be summarized as follows.

Suppose $f(u) = u + \frac{u^{p+1}}{p+1}$, where $p > 0$ is an integer, and p and $m(k)$ satisfy the following conditions:

A1. there exist positive constants A_1 and $r > \frac{p}{2}$ such that $m(k) \leqslant A_1|k|^r$ for $|k| \leqslant 1$;

A2. there exist positive constants A_2, A_3 and $s \geqslant 1$ such that $A_2|k|^s \leqslant m(k) \leqslant A_3|k|^s$ for $|k| \geqslant 1$;

A3. $m(k) \geqslant 0$ for all values of k;

A4. $m(k)$ is four times differentiable for all non-zero values of k, and for each $j \in \{0, 1, 2, 3, 4\}$ there exist positive constants B_1 and B_2 such that

$$\left| \left(\frac{d}{dk}\right)^j \left(\frac{m(k) - m(0)}{k}\right) \right| \leqslant B_1|k|^{-j} \quad \text{for } 0 < |k| \leqslant 1 \tag{1.7a}$$

and

$$\left| \left(\frac{d}{dk}\right)^j \left(\frac{\sqrt{m(k)}}{k^{\frac{s}{2}}}\right) \right| \leqslant B_2|k|^{-j} \quad \text{for } |k| \geqslant 1. \tag{1.7b}$$

Then we prove below in Theorem 2.2 and Corollaries 2.3–2.5 that for every $q > 0$ there exists a non-empty set of G_q consisting of solitary-wave profile functions g having positive wavespeeds and satisfying

$$\int \left[\frac{g^2}{2} + \frac{g^{p+2}}{(p+1)(p+2)}\right] dx = q,$$

and for every $\varepsilon > 0$ and $g \in G_q$ there exists a $\delta > 0$ such that if

$$||u_0 - g||_{\frac{s}{2}} < \delta,$$

then the solution $u(\cdot, t)$ of Eq. (1.1) with $u(x, 0) = u_0$ satisfies

$$\inf_{g \in G_q} ||u(\cdot, t) - g||_{\frac{s}{2}} < \varepsilon$$

for all values of t. (Here $|| \cdot ||_{\frac{s}{2}}$ denotes the norm in the L^2-based Sobolev space $H^{\frac{s}{2}}(\mathbb{R})$.)

We remark that (1.7a) is satisfied if the derivatives of $m(k)$ up to order five are bounded on $(0, K]$ for some $K > 0$, and conditions A1–A4 are satisfied, for example, if $m(k) = a_1|k|^{b_1} + a_2|k|^{b_2} + \cdots + a_n|k|^{b_n}$ where $a_1, \ldots, a_n > 0$; $1 \leqslant b_1 < b_2 < \cdots < b_n$, and $p < 2b_1$. In particular, Theorem 2.2 applies to the generalized BBM equation (1.5) in which $m(k) = k^2$, when $p < 4$.

If condition A1 is replaced by the condition that $m(k)$ be a non-decreasing function of $|k|$, then for any integer $p > 0$, there exists a $q_{\geqslant} 0$ such that the above existence and stability result holds for all $q > q_0$. Moreover, if $f(u) = \frac{u^{p+1}}{p+1}$, then condition A1 can be dropped, so that for every positive integer p, G_q exists for all $q > 0$ if $m(k)$ satisfies conditions A2–A4. If p is odd, G_q also exists for all $q < 0$.

We will use the method of concentration compactness to establish the existence and stability result when $m(k)$ satisfies conditions A1–A4 in Section 2. Our variational problem bears similarities to that in [11], in which the existence of solitary-wave solutions of Benjamin-type equations is studied, and we adapt some ideas of theirs in dealing with vanishing and dichotomy. Our assumptions (1.7a) and (1.7b) on $m(k)$, like those in [1], are a result of resorting to Theorem 35 of [12], which provides commutation estimates for the associated Fourier multiplier operator M. In Section 3, we prove an existence and stability result for the case when $m(k)$ is a non-decreasing function of $|k|$, p is a an arbitrary positive integer, and $m(k)$ satisfies conditions A2–A4. Then we apply our result to the generalized BBM equation (1.5). We recover the above-mentioned stability results of Souganidis and Strauss, except that for $p > 4$ our method fails to apply to solitary waves with wavespeeds c in the range $c_r < c < \frac{p}{4}$. Finally, in Section 4, we discuss the situation when $f(u) = \frac{u^{p+1}}{p+1}$, and give an example of how, by using techniques from [3,11], our method may be applied in cases where the Fourier multiplier operator M has a symbol that is not everywhere positive.

The notation used in this paper is the standard notation used in the literature on partial differential equations. The set of all real numbers is denoted by $\mathbb{R}$ and that of all natural numbers by $\mathbb{N}$. The support of a function f is denoted by $\operatorname{supp} f$, and B_R denotes the ball of radius R in $\mathbb{R}$ centered at zero. If A and B are two subsets of $\mathbb{R}$, the distance between them is defined to be $\inf\{|x-y|, x \in A$ and $y \in B\}$ and is denoted by $\operatorname{dist}(A, B)$. If X is any Banach space and $T > 0$, then $C(0, T; X)$ is the space of continuous mappings of the interval $[0, T]$ into X. The value $T = \infty$ is allowed in this definition. If k is a positive integer, $C^k(0, T; X)$ is the subspace of $C(0, T; X)$ of functions whose first k derivatives also lie in $C(0, T; X)$; also $C^\infty(0, T; X) = \bigcap_{k=1}^{\infty} C^k(0, T; X)$. We use $|\cdot|_p$ for the norm in $L^p(\mathbb{R})$ and $||\cdot||_s$ for the norm in the L^2-based Sobolev space $H^s(\mathbb{R})$. An integral over the set of all real numbers is denoted by $\int$, while an integral over a subset of $\mathbb{R}$, say $[a, b]$, is denoted by $\int_{[a,b]}$. The Gamma function $\Gamma(s)$ is defined for any s with $\operatorname{Re} s > 0$ by

$$\Gamma(s) = \int_0^\infty e^{-t} t^{s-1} \, dt.$$

Finally, if α is a quantity depending on a small parameter $\varepsilon > 0$, we write $\alpha \sim \varepsilon$ if $\lim_{\varepsilon \to 0} \frac{\alpha}{\varepsilon}$ exists and is non-zero, $\alpha = o(\varepsilon)$ if $\lim_{\varepsilon \to 0} \frac{\alpha}{\varepsilon} = 0$, and $\alpha = O(\varepsilon)$ if there exists a constant C such that $|\alpha| \leqslant C\varepsilon$ for sufficiently small ε.

2. Stability theory for $p < 2r$

In this section, we establish the existence of the stable set G_q consisting of solitary-wave profile functions for any $q > 0$, assuming that $f(u) = u + \frac{u^{p+1}}{p+1}$ and that A1–A4 hold for p and $m(k)$.

For information on well-posedness of Eq. (1.1), we refer readers to [2]. Here we merely state the following theorem which is a consequence of Theorem 2 of [2].

Theorem 2.1. *For any* $s > 0$, *if* $u_0 \in H^{\frac{s}{2}}(\mathbb{R})$, *then there exists a unique global solution* $u = u(x,t)$ *of* (1.1) *with* $u(x,0) = u_0$ *such that for* $0 < t < \infty$, *the map* $t \mapsto u(x,t)$ *lies in* $C^\infty(0, \infty; H^{\frac{s}{2}}(\mathbb{R}))$.

Several invariants of Eq. (1.1) can be established by standard arguments. In particular, it is easy to show that if $u(x,t)$ is the solution described in Theorem 2.1, then the functionals defined by

$$E(u) = \int (u^2 + uMu)\,dx \tag{2.1}$$

and

$$Q(u) = \int F(u)\,dx,$$

where $F'(x) = f(x)$ and $F(0) = 0$, satisfy $E(u) = E(u_0)$ and $Q(u) = Q(u_0)$ for all $t \in \mathbb{R}$. In this section, since $f(u) = u + \frac{u^{p+1}}{p+1}$, we define

$$Q(u) = \int \left[\frac{u^2}{2} + \frac{u^{p+2}}{(p+1)(p+2)}\right] dx. \tag{2.2}$$

A solitary-wave solution to Eq. (1.1) is a solution of the form $u = \phi_c(x - ct)$. The wave profile function ϕ_c then needs to satisfy

$$f(\phi_c) = c(\phi_c + M\phi_c). \tag{2.3}$$

Eq. (2.3) can be obtained by substituting $u = \phi_c(x - ct)$ into (1.1) and observing that the resulting equation is true for all values of x and t.

Next we define a variational problem whose Euler–Lagrange equation corresponds to (2.3). For any $q > 0$, define

$$I_q = \inf\{E(u) \mid u \in H^{\frac{s}{2}}(\mathbb{R}) \text{ and } Q(u) = q\}$$

and

$$G_q = \{u \in H^{\frac{s}{2}}(\mathbb{R}) \mid Q(u) = q \text{ and } E(u) = I_q\},$$

i.e., G_q is the set of minimizers of I_q. A minimizing sequence for I_q is any sequence $\{u_n\}$ in $H^{\frac{s}{2}}(\mathbb{R})$ that has the property

$$Q(u_n) = q \quad \text{for all } n$$

and

$$\lim_{n\to\infty} E(u_n) = I_q.$$

We can now state our main existence and stability theorem.

Theorem 2.2. *Suppose the assumptions* A1–A4 *are satisfied by* p *and* $m(k)$. *Then* G_q *is non-empty for every* $q > 0$. *Moreover, for every minimizing sequence* $\{u_n\}$, *there exists a sequence of real numbers* $\{y_n\}$, *such that* $\{u_n(\cdot + y_n)\}$ *has a subsequence that converges in* $H^{\frac{s}{2}}(\mathbb{R})$ *to an element* $g \in G_q$.

Before proving Theorem 2.2, let us see how it implies the existence and stability of solitary-wave solutions. The arguments which follow are standard, and can be found in, e.g., [1,10,13].

Corollary 2.3. *If* $\{u_n\}$ *is a minimizing sequence for* I_q, *then* $u_n \to G_q$ *in* $H^{\frac{s}{2}}(\mathbb{R})$, *i.e.*,

$$\lim_{n\to\infty} \inf_{g\in G_q} ||u_n - g||_{\frac{s}{2}} = 0.$$

Proof. We first show

$$\lim_{n\to\infty} \inf_{\substack{g\in G_q \\ y\in\mathbb{R}}} ||u_n(\cdot + y) - g||_{\frac{s}{2}} = 0.$$

If this is not true, then for some $\varepsilon > 0$, there exists a subsequence $\{u_{n_k}\}$, such that

$$\inf_{\substack{g\in G_q \\ y\in\mathbb{R}}} ||u_{n_k}(\cdot + y) - g||_{\frac{s}{2}} \geqslant \varepsilon.$$

But $\{u_{n_k}\}$ is itself a minimizing sequence, so the above inequality contradicts Theorem 2.2.

Now for any $y\in\mathbb{R}$ and $g \in G_q$,

$$||u_n(\cdot + y) - g||_{\frac{s}{2}} = ||u_n - g(\cdot - y)||_{\frac{s}{2}}.$$

Since $g(\cdot - y)$ is also in G_q, our equality follows. □

Corollary 2.4 (existence of solitary waves). G_q *consists of solitary wave profiles.*

Proof. We must show that elements of G_q are solutions of (2.3) for some c. If $g \in G_q$, then by the Lagrange multiplier principle (see, e.g., [19]), there exists a $\lambda \in \mathbb{R}$ such that

$$\delta E(g) = \lambda \delta Q(g),$$

where $\delta E(g)$ and $\delta Q(g)$ are the Frechet derivatives of E and Q at g. For any $\phi \in H^{\frac{s}{2}}(\mathbb{R})$,

$$\delta E(g)\phi = \lim_{\varepsilon\to 0} \frac{E(g+\varepsilon\phi) - E(g)}{\varepsilon}$$

and

$$\delta Q(g)\phi = \lim_{\varepsilon\to 0} \frac{Q(g+\varepsilon\phi) - Q(g)}{\varepsilon}.$$

Substituting (2.1) and (2.2) into the above equations and simplifying, we get

$$\delta E(g)\phi = \int (2g + 2Mg)\phi \, dx$$

and

$$\delta Q(g)\phi = \int \left(g + \frac{g^{p+1}}{p+1} \right) \phi \, dx.$$

Hence

$$\int (2g + 2Mg)\phi \, dx = \lambda \int \left(g + \frac{g^{p+1}}{p+1} \right) \phi \, dx$$

for all $\phi \in H^{\frac{s}{2}}(\mathbb{R})$. It follows that

$$2g + 2Mg = \lambda \left(g + \frac{g^{p+1}}{p+1} \right).$$

We see then that g is a solitary-wave profile function with wave speed $\frac{2}{\lambda}$.

To see λ is positive, we apply $\delta E(g)$ and $\delta Q(g)$ on g and get

$$\int (2g + 2Mg) g \, dx = \lambda \int \left(g^2 + \frac{g^{p+2}}{p+1} \right) dx.$$

We see that the left-hand side is equal to I_q. It then follows Lemmas 2.8 and 2.10 that $\lambda > 0$. $\square$

Corollary 2.5 (stability of solitary waves). *G_q is a stable set in the following sense: for every $\varepsilon > 0$ and $g \in G_q$, there exists a $\delta > 0$ such that if*

$$||u_0 - g||_{\frac{s}{2}} < \delta,$$

then the solution $u(x,t)$ of (1.1) *with $u(x,0) = u_0$ satisfies*

$$\inf_{g \in G_q} ||u(\cdot,t) - g||_{\frac{s}{2}} < \varepsilon$$

for all $t \in \mathbb{R}$.

Proof. Suppose the theorem is false; then there exist a $g_0 \in G_q$ and $\varepsilon_0 > 0$, such that for every $n \in \mathbb{N}$, we can find $\phi_n \in H^{\frac{s}{2}}(\mathbb{R})$ and $t_n \in \mathbb{R}$ such that

$$||\phi_n - g_0||_{\frac{s}{2}} < \frac{1}{n}$$

and

$$\inf_{g \in G_q} ||u_n(\cdot, t_n) - g||_{\frac{s}{2}} \geqslant \varepsilon_0,$$

where $u_n(\cdot, t_n)$ is the solution of (1.1) with $u_n(\cdot, 0) = \phi_n$. Since $\phi_n \to g_0$ in $H^{\frac{s}{2}}(\mathbb{R})$, then $Q(\phi_n) \to q$ and $E(\phi_n) \to I_q$. Hence $Q(u_n(\cdot, t_n)) \to q$ and $E(u_n(\cdot, t_n)) \to I_q$. Now choose $\alpha_n \in \mathbb{R}$ such that $Q(\alpha_n u_n(\cdot, t_n)) = q$; then $\alpha_n \to 1$. Thus

$$\lim_{n \to \infty} E(\alpha_n u_n(\cdot, t_n)) = \lim_{n \to \infty} \alpha_n^2 E(u_n(\cdot, t_n)) = I_q,$$

i.e., $\{\alpha_n u_n(\cdot, t_n)\}$ is a minimizing sequence of I_q. Therefore, by Corollary 2.3, for sufficiently large n there exists $g_n \in G_q$ such that

$$||\alpha_n u_n(\cdot, t_n) - g_n||_{\frac{s}{2}} < \frac{\varepsilon_0}{2}.$$

So

$$\begin{aligned}\varepsilon_0 \leqslant ||u_n(\cdot, t_n) - g_n||_{\frac{s}{2}} &\leqslant ||u_n(\cdot, t_n) - \alpha_n u_n(\cdot, t_n)||_{\frac{s}{2}} + ||\alpha_n u_n(\cdot, t_n) - g_n||_{\frac{s}{2}} \\ &< |1 - \alpha_n| ||u_n(\cdot, t_n)||_{\frac{s}{2}} + \frac{\varepsilon_0}{2}.\end{aligned}$$

Contradiction is then reached when we let $n \to \infty$. $\square$

We now proceed to prove Theorem 2.2 using the method of concentration compactness. Key to the proof is the following lemma of Lions.

Lemma 2.6 (Lions [17]). *Let $\{\rho_n\}$ be a sequence in $L^1(\mathbb{R})$ satisfying*:

$$\rho_n \geqslant 0 \text{ on } \mathbb{R} \quad \text{and} \quad \int \rho_n \, dx = \mu,$$

where $\mu > 0$ is fixed. Then there exists a subsequence $\{\rho_{n_k}\}$ with one of the three following properties:

(1) (compactness). *There exists a sequence $y_k \in \mathbb{R}$ such that for every $\varepsilon > 0$, there exists $R < \infty$ satisfying for all $k \in \mathbb{N}$*:

$$\int_{y_k + B_R} \rho_{n_k}(x) \, dx \geqslant \mu - \varepsilon,$$

(2) (vanishing). *For all $R < +\infty$,*

$$\lim_{k \to \infty} \sup_{y \in \mathbb{R}} \int_{y + B_R} \rho_{n_k}(x) \, dx = 0$$

or

(3) (dichotomy). *There exists $\bar{\mu} \in (0, \mu)$ such that for every $\varepsilon > 0$, there exist $k_0 \geqslant 1$ and two sequences of positive functions $\rho_k^{(1)}, \rho_k^{(2)} \in L^1(\mathbb{R})$ satisfying for $k \geqslant k_0$*:

$$|\rho_{n_k} - (\rho_k^{(1)} + \rho_k^{(2)})|_1 \leqslant \varepsilon,$$

$$\left| \int \rho_k^{(1)} \, dx - \bar{\mu} \right| \leqslant \varepsilon,$$

$$\left| \int \rho_k^{(2)} \, dx - (\mu - \bar{\mu}) \right| \leqslant \varepsilon,$$

$$\operatorname{dist}(\operatorname{supp} \rho_k^{(1)}, \operatorname{supp} \rho_k^{(2)}) \to \infty.$$

Remark. In the above Lemma, as remarked in [11], the condition $\int \rho_n(x)\,dx = \mu$ can be replaced by $\int \rho_n(x)\,dx = \mu_n$ where $\mu_n \to \mu > 0$.

Before applying Lemma 2.6, we need some preparation.

Lemma 2.7. *If $\{u_n\}$ is a minimizing sequence, then there exist $M > 0$ and $N > 0$ such that $N \leqslant ||u_n||_{\frac{s}{2}} \leqslant M$ for all n.*

Proof. By assumptions A2 and A3 on $m(k)$, there exist positive constants C_1 and C_2 such that

$$C_1(1+k^2)^{\frac{s}{2}} \leqslant 1 + m(k) \leqslant C_2(1+k^2)^{\frac{s}{2}} \quad \text{for all } k \in \mathbb{R}.$$

So for any $u \in H^{\frac{s}{2}}(\mathbb{R})$,

$$C_1||u||^2_{\frac{s}{2}} \leqslant E(u) = \int [1+m(k)]|\hat{u}(k)|^2\,dk \leqslant C_2||u||^2_{\frac{s}{2}}. \tag{2.4}$$

Since $\lim_{n\to\infty} E(u_n) = I_q$ and $C_1||u_n||^2_{\frac{s}{2}} \leqslant E(u_n)$, $\{u_n\}$ is bounded in $H^{\frac{s}{2}}(\mathbb{R})$.

To bound $||u_n||_{\frac{s}{2}}$ from below, we write

$$\int \left[\frac{u_n^2}{2} + \frac{u_n^{p+2}}{(p+1)(p+2)}\right] dx = q.$$

So

$$\frac{1}{2}\int |u_n|^2\,dx + \frac{1}{(p+1)(p+2)}\int |u_n|^{p+2}\,dx \geqslant q,$$

hence

$$A||u_n||^2_{\frac{s}{2}} + B||u_n||^{p+2}_{\frac{s}{2}} \geqslant q,$$

where the Sobolev imbedding theorem has been used, and A and B denote positive constants independent of n. We then have

$$||u_n||^2_{\frac{s}{2}}(A + B||u_n||^p_{\frac{s}{2}}) \geqslant q.$$

Therefore

$$||u_n||^2_{\frac{s}{2}} \geqslant \frac{q}{(A+BM^p)},$$

so the desired N exists. $\square$

Lemma 2.8. $I_q > 0$.

Proof. By Lemma 2.7 and (2.4),

$$I_q = \lim_{n\to\infty} E(u_n) \geqslant \lim_{n\to\infty} C_1 \|u_n\|^2_{\frac{s}{2}} \geqslant C_1 N^2 > 0. \qquad \square$$

Lemma 2.9. *If $q_2 > q_1 > 0$, then $I_{q_2} \geqslant I_{q_1}$.*

Proof. For any $\varepsilon > 0$, there exists a function $\phi \in H^{\frac{s}{2}}(\mathbb{R})$ such that $Q(\phi) = q_2$ and $E(\phi) < I_{q_2} + \varepsilon$. Since $Q(a\phi)$ is a continuous function of $a \in \mathbb{R}$, then by the intermediate value theorem we can find $A \in (0,1)$ such that $Q(A\phi) = q_1$. Hence

$$I_{q_1} \leqslant E(A\phi) = A^2 E(\phi) < E(\phi) < I_{q_2} + \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, it follows that

$$I_{q_1} \leqslant I_{q_2}. \qquad \square$$

Lemma 2.10. *If $\{u_n\}$ is a minimizing sequence, then there exists a $P > 0$ such that*

$$\int u_n^{p+2}\, dx \geqslant P$$

for sufficiently large n.

Proof. Since $\{u_n\}$ is a minimizing sequence for I_q, it is also a minimizing sequence for $\bar{I}_q = \inf\{\bar{E}(u) | u \in H^{\frac{s}{2}}(\mathbb{R}) \text{ and } Q(u) = q\}$, where

$$\begin{aligned} \bar{E}(u) &= E(u) - 2Q(u) \\ &= \int \left[uMu - \frac{2}{(p+1)(p+2)} u^{p+2} \right] dx. \end{aligned}$$

Next we will show that $\bar{I}_q < 0$. To see this, let ϕ be a function such that $Q(\phi) = q$, $\int \phi^{p+2}\, dx > 0$, and $\hat{\phi}(k)$ is non-zero only in the set of values of k for which the inequality $m(k) \leqslant A_1 |k|^r$ of assumption A1 holds. (This can be done, for example, by letting $\phi(x) = \frac{a \sin \omega x}{\omega x}$, whose Fourier transform satisfies $\hat{\phi}(k) = \frac{a\pi}{\omega}$ for $|k| < \omega$ and $\hat{\phi}(k) = 0$ for $|k| > \omega$, and appropriately choosing a and ω.) For any $\theta > 0$, choose $\alpha > 0$ such that $\phi_\theta(x) = \alpha\phi(\theta x)$ satisfies $Q(\phi_\theta(x)) = q$. Then

$$\int \left[\frac{1}{2} \alpha^2 \phi^2(\theta x) + \frac{\alpha^{p+2}}{(p+1)(p+2)} \phi^{p+2}(\theta x) \right] dx = q,$$

i.e.,

$$\alpha^2 \int \frac{1}{2} \phi^2(y)\, dy + \alpha^{p+2} \int \frac{\phi^{p+2}(y)}{(p+1)(p+2)}\, dy = \theta q.$$

Now

$$\begin{aligned}\bar{E}(\phi_\theta(x)) &= \int m(x)|\hat{\phi}_\theta(x)|^2\,dx - \frac{2\alpha^{p+2}}{(p+1)(p+2)}\int \phi^{p+2}(\theta x)\,dx\\ &= \frac{\alpha^2}{\theta}\int m(y\theta)|\hat{\phi}(y)|^2\,dy - \frac{2\alpha^{p+2}}{(p+1)(p+2)\theta}\int \phi^{p+2}(y)\,dy\\ &\leqslant \frac{A_1\alpha^2}{\theta^{1-r}}\int |y|^r|\hat{\phi}(y)|^2\,dy - \frac{2\alpha^{p+2}}{(p+1)(p+2)\theta}\int \phi^{p+2}(y)\,dy.\end{aligned}$$

Letting $\theta \to 0$, we see $\alpha^2 \sim \theta$; so $\frac{\alpha^2}{\theta^{1-r}} \sim \theta^r$ and $\frac{\alpha^{p+2}}{\theta} \sim \theta^{\frac{p}{2}}$. Since $p < \frac{r}{2}$, θ^r is a higher order infinitesimal than $\theta^{\frac{p}{2}}$. So $\bar{E}(\phi_\theta(x))$ can be made less than 0 for sufficiently small θ. Hence $\bar{I}_q < 0$.

The proof of the Lemma now follows by contradiction. Indeed, suppose the conclusion of the Lemma to be false. Then

$$\liminf \int u_n^{p+2}\,dx \leqslant 0,$$

and consequently

$$\begin{aligned}\bar{I}_q &= \lim_{n\to\infty}\int\left[m(x)|\hat{u}_n(x)|^2 - \frac{2}{(p+1)(p+2)}u_n^{p+2}\right]dx\\ &\geqslant \limsup\left(-\frac{2}{(p+1)(p+2)}\int u_n^{p+2}\,dx\right)\\ &= -\liminf \frac{2}{(p+1)(p+2)}\int u_n^{p+2}\,dx\\ &\geqslant 0,\end{aligned}$$

which contradicts the result of the preceding paragraph. $\square$

Lemma 2.11. *For every $q_1 > 0$ and every $q_2 > 0$, $I_{q_1+q_2} < I_{q_1} + I_{q_2}$.*

Proof. We first show that for $\theta > 1$ and $q > 0$, $I_{\theta q} < \theta I_q$. Let $\{\phi_n\}$ be a minimizing sequence for I_q. Choose $\alpha_n > 0$ such that $Q(\alpha_n\phi_n) = \theta q$; then

$$\alpha_n^2\int \frac{1}{2}\phi_n^2\,dx + \alpha_n^{p+2}\int \frac{\phi_n^{p+2}}{(p+2)(p+1)}\,dx = \theta q. \tag{2.5}$$

Since

$$\int \frac{1}{2}\phi_n^2\,dx + \int \frac{\phi_n^{p+2}}{(p+1)(p+2)}\,dx = q, \tag{2.6}$$

we have

$$\alpha_n^2 = \theta - \frac{\alpha_n^2(\alpha_n^p - 1)}{q(p+1)(p+2)} \int \phi_n^{p+2}\, dx.$$

Thus

$$\begin{aligned} I_{\theta q} &\leqslant E(\alpha_n \phi_n) = \alpha_n^2 E(\phi_n) \\ &= \left[\theta - \frac{\alpha_n^2(\alpha_n^p - 1)}{q(p+1)(p+2)} \int \phi_n^{p+2}\, dx\right] E(\phi_n). \end{aligned} \tag{2.7}$$

Since $\{\phi_n\}$ is a minimizing sequence for I_q, then by Lemma 2.10, $\int \phi_n^{p+2}\, dx \geqslant P$ for some $P > 0$ when n is sufficiently large. We see, from (2.5) and (2.6), that there exists $\varepsilon > 0$ such that $\alpha_n \geqslant 1 + \varepsilon$ for sufficiently large n. Hence by (2.7) there exists an $A > 0$ such that, again for sufficiently large n,

$$I_{\theta q} \leqslant (\theta - A) E(\phi_n).$$

Letting $n \to \infty$ in the above inequality we obtain

$$I_{\theta q} \leqslant (\theta - A) I_q < \theta I_q.$$

Now, assuming without loss of generality that $q_1 \geqslant q_2$, we can use the result of the preceding paragraph to write

$$\begin{aligned} I_{q_1+q_2} &= I_{q_1(1+\frac{q_2}{q_1})} \\ &< \left(1 + \frac{q_2}{q_1}\right) I_{q_1} \\ &\leqslant I_{q_1} + \frac{q_2}{q_1}\left(\frac{q_1}{q_2} I_{q_2}\right) \\ &= I_{q_1} + I_{q_2}. \qquad \square \end{aligned}$$

The following lemma is from [4] (see Lemma 3.5 in [4] and Lemma 4.2 in [1]).

Lemma 2.12. *We can write $M = M_1 + M_2^2$ where M_1 and M_2 are self-adjoint operators on $H^{\frac{s}{2}}(\mathbb{R})$ with the following properties*:

(i) *there exists a positive constant A such that for every f in $H^{\frac{s}{2}}(\mathbb{R})$ and every θ which is in $L^\infty(\mathbb{R})$ and has derivatives θ' in $L^\infty(\mathbb{R})$,*

$$|[M_1, \theta] f|_2 \leqslant A |\theta'|_\infty |f|_2,$$

where $[M_1, \theta] f$ is defined to be $M_1(\theta f) - \theta(M_1 f)$.

(ii) *there exists a positive constant A such that for every f in $H^{\frac{s}{2}}(\mathbb{R})$ and every θ which is in $L^\infty(\mathbb{R})$ and derivative up to order $S = [\frac{s}{2}] + 1$ in $L^\infty(\mathbb{R})$,*

$$|[M_2, \theta] f|_2 \leqslant A |\theta'|_\infty |f|_2,$$

where $[\frac{s}{2}]$ denotes the greatest integer less than or equal to $\frac{s}{2}$.

Proof. Since $C_0^\infty(\mathbb{R})$ is dense in $H^{\frac{s}{2}}(\mathbb{R})$, it suffices to prove this lemma for $f\in C_0^\infty(\mathbb{R})$.

Choose $\chi(k)\in C_0^\infty(\mathbb{R})$ such that $\chi(k)=1$ for $|k|<1$ and $\chi(k)=0$ for $|k|>2$. Let $m_1(k)=\chi(k)m(k)$, $m_2(k)=\sqrt{(1-\chi(k))m(k)}$. Define M_1 and M_2 by $\widehat{M_1u}(k)=m_1(k)\hat{u}(k)$ and $\widehat{M_2u}(k)=m_2(k)\hat{u}(k)$; then $M=M_1+M_2^2$.

Let $\tilde{m}_1(k)=m_1(k)-m_1(0)=\chi(k)(m(k)-m(0))$. Define $\widehat{\tilde{M}_1u}(k)=\tilde{m}_1(k)\hat{u}(k)$ and write $M_1=\frac{d}{dx}T_1$; the symbol of T_1 is then given by

$$\sigma_1(k)=\frac{\chi(k)(m(k)-m(0))}{ik}.$$

It follows from (1.7a) that

$$\sup_{k\in\mathbb{R}}|k|^j\left|\left(\frac{d}{dk}\right)^j\sigma_1(k)\right|<\infty,$$

for all $j\in\{0,1,2,3,4\}$. Now Theorem 35 of [12] implies that there exists a positive constant C such that

$$|[T_1,\theta]f'|_2\leqslant C|\theta'|_\infty|f|_2.$$

Hence

$$\begin{aligned}
|[M_1,\theta]f|_2&=|[\tilde{M}_1,\theta]f|_2\\
&=\left|T_1\frac{d}{dx}(\theta f)-\theta T_1\left(\frac{df}{dx}\right)\right|_2\\
&\leqslant|T_1(\theta'f)|_2+|[T_1,\theta]f'|_2\\
&\leqslant||T_1|||\theta'|_\infty|f|_2+C|\theta'|_\infty|f|_2\\
&=A|\theta'|_\infty|f|_2,
\end{aligned}$$

where $A=C+||T_1||$.

Write $M_2=(\frac{d}{dx})^ST_2$; the symbol of T_2 is then given by

$$\sigma_2(k)=\frac{m_2(k)}{(ik)^S}.$$

It follows from (1.7b) that

$$\sup_{k\in\mathbb{R}}|k|^j\left|\left(\frac{d}{dk}\right)^j\sigma_2(k)\right|<\infty$$

for all $j\in\{0,1,2,3,4\}$. Again by Theorem 35 of [12], there exists $C>0$ such that

$$|[T_2,\theta]f'|_2\leqslant C|\theta'|_\infty|f|_2.$$

Hence

$$\begin{aligned}|[M_2,\theta]f|_2 &= \left|T_2\left(\frac{d^S(\theta f)}{dx^S}\right) - \theta T_2\left(\frac{d^S f}{dx^S}\right)\right|_2 \\ &= \left|[T_2,\theta]\left(\frac{d^S f}{dx^S}\right) + T_2\left(\sum_{i=1}^{S} a_i^S \frac{d^i\theta}{dx^i}\frac{d^{S-i}f}{dx^{S-i}}\right)\right|_2 \\ &\leqslant A\left(\sum_{i=1}^{S}\left|\frac{d^i\theta}{dx^i}\right|_\infty\right)\left(\sup_{0\leqslant i\leqslant S-1}\left|\frac{d^i f}{dx^i}\right|_2\right) \\ &\leqslant A\left(\sum_{i=1}^{S}\left|\frac{d^i\theta}{dx^i}\right|_\infty\right)||f||_{\frac{s}{2}},\end{aligned}$$

where a_i^S are constants which come from Liebniz' rule, and A is a positive constant independent of θ and f. The proof is then completed. $\square$

Let $\rho_n = u_n^2 + (M_2 u_n)^2$ and $\mu_n = \int \rho_n\, dx$ Then there exist $D_1,\ D_2 > 0$ such that

$$D_1||u_n||_{\frac{s}{2}}^2 \leqslant \int \rho_n\, dx \leqslant D_2||u_n||_{\frac{s}{2}}^2.$$

By Lemma 2.7, there exist a real number $\mu > 0$ and a subsequence of $\{\rho_n\}$, still denoted by $\{\rho_n\}$, such that $\int \rho_n\, dx \to \mu$. Now by Lemma 2.6 and the remark following it, there exists a subsequence of $\{\rho_n\}$, still denoted by $\{\rho_n\}$, for which vanishing, dichotomy or compactness holds. In what follows, we will eliminate vanishing and dichotomy, and we will see that compactness then leads to Theorem 2.2.

To eliminate the case of vanishing, we need the following lemma from [11].

Lemma 2.13. *Let $1 \leqslant p < \infty$ and $1 \leqslant q < \infty$. If $\{u_n\}$ is bounded in $L^q(\mathbb{R})$, $\{u_n'\}$ is bounded in $L^p(\mathbb{R})$, and for some $R > 0$,*

$$\lim_{n\to\infty} \sup_{y\in\mathbb{R}} \int_{|x-y|\leqslant R} |u_n(x)|^q\, dx = 0,$$

then for all $r > q$, $u_n \to 0$ in $L^r(\mathbb{R})$.

Lemma 2.14. *Vanishing does not occur.*

Proof. If it does, then for every $R > 0$,

$$\lim_{n\to\infty} \sup_{y\in\mathbb{R}} \int_{|x-y|\leqslant R} \rho_n(x)\, dx = 0,$$

thus

$$\lim_{n\to\infty} \sup_{y\in\mathbb{R}} \int_{|x-y|\leqslant R} u_n^2(x)\, dx = 0.$$

Since $\{u_n'\}$ is obviously bounded in $L^2(\mathbb{R})$, by Lemma 2.13, $u_n \to 0$ in $L^{p+2}(\mathbb{R})$, and this contradicts Lemma 2.10. $\square$

Lemma 2.15. *Assume the dichotomy alternative of Lemma* 2.6 *holds for* ρ_n. *Then for each* $\varepsilon > 0$ *there is a subsequence of* $\{u_n(x)\}$, *still denoted by* $\{u_n(x)\}$, *a real number* $\bar{q} = \bar{q}(\varepsilon)$, *a natural number* n_0, *and two sequences of functions* $\{u_n^{(1)}\}$ *and* $\{u_n^{(2)}\}$ *in* $H^{\frac{s}{2}}(\mathbb{R})$ *satisfying* $u_n = u_n^{(1)} + u_n^{(2)}$ *for all n and for* $n \geqslant n_0$:

$$Q(u_n^{(1)}) - \bar{q} = O(\varepsilon),$$

$$Q(u_n^{(2)}) - (q - \bar{q}) = O(\varepsilon),$$

$$E(u_n) = E(u_n^{(1)}) + E(u_n^{(2)}) + O(\varepsilon),$$

where the constants implied in the notation $O(\varepsilon)$ *can be chosen independently of n as well as* ε. *Furthermore*,

$$E(u_n^1) \geqslant \bar{\mu} + O(\varepsilon) \tag{2.8a}$$

and

$$E(u_n^2) \geqslant \mu - \bar{\mu} + O(\varepsilon), \tag{2.8b}$$

where $\bar{\mu}$ *is as defined in Lemma* 2.6.

Proof. We follow the general lines of the proof of Theorem 2.5 in [11]. By assumption, for every $\varepsilon > 0$, we can find a number k_0 and sequences of positive functions $\{\rho_n^{(1)}\}$ and $\{\rho_n^{(2)}\}$ in $L^1(\mathbb{R})$ satisfying for $n \geqslant k_0$:

$$\left| \int \rho_n^{(1)}\, dx - \bar{\mu} \right| \leqslant \varepsilon,$$

$$\left| \int \rho_n^{(2)}\, dx - (\mu - \bar{\mu}) \right| \leqslant \varepsilon, \quad \text{and}$$

$$|\rho_n - (\rho_n^{(1)} + \rho_n^{(2)})|_1 \leqslant \varepsilon.$$

Moreover, without loss of generality (see the proof of Lemma 2.6 in [17]), we may assume that $\rho_n^{(1)}$ and $\rho_n^{(2)}$ satisfy

$$\operatorname{supp} \rho_n^{(1)} \subset (y_n - R_n, y_n + R_n),$$

$$\operatorname{supp} \rho_n^{(2)} \subset (-\infty, y_n - 4R_n) \cup (y_n + 4R_n, \infty),$$

where $y_n \in \mathbb{R}$ and $R_n \to \infty$. We then have

$$\int_{R_n \leqslant |x - y_n| \leqslant 4R_n} \rho_n\, dx \leqslant \varepsilon,$$

hence

$$\int_{R_n \leqslant |x-y_n| \leqslant 4R_n} [u_n^2 + (M_2 u_n)^2]\, dx \leqslant \varepsilon.$$

Choose $\zeta, \phi \in C^\infty(\mathbb{R})$ such that $0 \leqslant \zeta(x), \phi(x) \leqslant 1$ for all x; $\zeta(x) = 1$ if $|x| \leqslant 2$; $\zeta(x) = 0$ if $|x| \geqslant 3$; $\phi(x) = 0$ if $|x| \leqslant 2$; $\phi(x) = 1$ if $|x| \geqslant 3$; and $\zeta^2 + \phi^2 = 1$ for all $x \in \mathbb{R}$. Define $\eta \in C^\infty(\mathbb{R})$ so that $0 \leqslant \eta \leqslant 1$, $\eta(x) = 1$ for $2 \leqslant |x| \leqslant 3$, and $\eta(x) = 0$ for $|x| \leqslant 1$ and $|x| \geqslant 4$. Let $\zeta_n(x) = \zeta(\frac{x-y_n}{R_n})$, $\phi_n = \phi(\frac{x-y_n}{R_n})$, and $\eta_n(x) = \eta(\frac{x-y_n}{R_n})$; and define $u_n^{(1)} = \zeta_n u_n$, $u_n^{(2)} = \phi_n u_n$, and $w_n = \eta_n u_n$.

Since $Q(u_n^{(1)})$ is bounded, there exists a subsequence of $u_n^{(1)}$, still denoted by $u_n^{(1)}$, and a $\bar{q} = \bar{q}(\varepsilon)$ such that $Q(u_n^{(1)}) \to \bar{q}$. Then, for sufficiently large n,

$$Q(u_n^{(1)}) - \bar{q} = O(\varepsilon).$$

Now let

$$f(s) = \frac{s^2}{2} + \frac{s^{p+2}}{(p+1)(p+2)} \quad \text{for } s \in \mathbb{R},$$

and write

$$\begin{aligned} Q(u_n) &= \int_{|x-y_n| \leqslant 2R_n} f(u_n)\, dx + \int_{|x-y_n| \geqslant 3R_n} f(u_n)\, dx \\ &\quad + \int_{2R_n \leqslant |x-y_n| \leqslant 3R_n} f(u_n)\, dx \\ &= \int_{|x-y_n| \leqslant 2R_n} (u_n^{(1)})\, dx + \int_{|x-y_n| \geqslant 3R_n} f(u_n^{(2)})\, dx \\ &\quad + \int_{2R_n \leqslant |x-y_n| \leqslant 3R_n} f(u_n)\, dx \\ &= Q(u_n^{(1)}) + Q(u_n^{(2)}) + \int_{2R_n \leqslant |x-y_n| \leqslant 3R_n} \\ &\quad \times [f(u_n) - f(u_n^{(1)}) - f(u_n^{(2)})]\, dx. \end{aligned} \tag{2.9}$$

We now claim that the last integral on the right-hand side of the preceding equation is $O(\varepsilon)$. To see this, first note that

$$\begin{aligned} &\left| \int_{2R_n \leqslant |x-y_n| \leqslant 3R_n} [f(u_n) - f(u_n^{(1)}) - f(u_n^{(2)})]\, dx \right| \leqslant C \int [|w_n|^2 + |w_n|^{p+2}]\, dx \\ &\quad \leqslant C(||w_n||_{\frac{s}{2}}^2 + ||w_n||_{\frac{s}{2}}^{p+2}). \end{aligned} \tag{2.10}$$

Therefore it suffices to show that $||w_n||^2_{\frac{s}{2}} = O(\varepsilon)$. To see this, write

$$\begin{aligned} D_1||w_n||^2_{\frac{s}{2}} &\leqslant \int (w_n^2 + w_n M_2^2 w_n)\,dx \\ &= \int [w_n^2 + (M_2 w_n)^2]\,dx \\ &= \int [(\eta_n u_n)^2 + (M_2(\eta_n u_n))^2]\,dx. \end{aligned}$$

The first term is small since

$$\int (\eta_n u_n)^2 dx = \int_{R_n \leqslant |x-y_n| \leqslant 4R_n} \eta_n^2 u_n^2\,dx \leqslant \varepsilon.$$

To estimate the second term, write

$$\begin{aligned} &\int [M_2(\eta_n u_n)]^2 dx \\ &\quad = \int ([M_2,\eta_n]u_n)^2\,dx + 2\int \eta_n M_2 u_n [M_2,\eta_n]u_n\,dx + \int \eta_n^2 (M_2 u_n)^2\,dx. \end{aligned} \tag{2.11}$$

The last integral on the right-hand side of (2.11) may be estimated as

$$\int \eta_n^2 (M_2 u_n)^2\,dx \leqslant \int_{R_n \leqslant |x-y_n| \leqslant 4R_n} (M_2 u_n)^2\,dx \leqslant \varepsilon.$$

For the other two terms on the right-hand side of (2.11), we apply Lemma 2.12, observing that $R_n \to \infty$ and $\{M_2 u_n\}$ is bounded in $L^2(\mathbb{R})$. It follows then from (2.11) that we can make $\int [M_2(\eta_n u_n)]^2\,dx$ a quantity of size $O(\varepsilon)$ for sufficiently large n. This completes our proof that $||w_n||_{\frac{s}{2}} = O(\varepsilon)$ for large n, and from (2.9) and (2.10) we can now conclude that

$$Q(u_n) = Q(u_n^{(1)}) + Q(u_n^{(2)}) + O(\varepsilon).$$

It then follows that

$$Q(u_n^{(2)}) = q - \bar{q} + O(\varepsilon).$$

To prove the assertion of the Lemma concerning $E(u_n)$, begin by writing

$$\begin{aligned} E(u_n^{(1)}) &= E(\zeta_n u_n) \\ &= \int [\zeta_n^2 u_n^2 + \zeta_n u_n M_1(\zeta_n u_n) + \zeta_n u_n M_2^2(\zeta_n u_n)]\,dx \\ &= \int \zeta_n^2 [u_n^2 + u_n M_1 u_n + (M_2 u_n)^2]\,dx + \int \zeta_n u_n [M_1,\zeta_n] u_n\,dx \\ &\quad + 2\int (\zeta_n M_2 u_n)[M_2,\zeta_n]u_n\,dx + \int ([M_2,\zeta_n]u_n)^2\,dx. \end{aligned}$$

In the last expression, all the terms that contain commutators are $O(\varepsilon)$. Hence

$$E(u_n^{(1)}) = \int \zeta_n^2[u_n^2 + u_n M_1 u_n + (M_2 u_n)^2]\, dx + O(\varepsilon).$$

Similarly

$$E(u_n^{(2)}) = \int \phi_n^2[u_n^2 + u_n M_1 u_n + (M_2 u_n)^2]\, dx + O(\varepsilon).$$

Adding the above equation then gives

$$E(u_n) = E(u_n^{(1)}) + E(u_n^{(2)}) + O(\varepsilon)$$

as desired It remains to prove (2.8). For (2.8a), we write

$$\begin{aligned} E(u_n^{(1)}) &\geqslant \int [(u_n^{(1)})^2 + (M_2 u_n^{(1)})^2]\, dx \\ &= \int [(\zeta_n u_n)^2 + (M_2(\zeta_n u_n))^2]\, dx \\ &= O(\varepsilon) + \int \zeta_n^2[u_n^2 + (M_2 u_n)^2]\, dx \\ &= O(\varepsilon) + \int \zeta_n^2 \rho_n\, dx \\ &= O(\varepsilon) + \int_{|x-y_n| \leqslant R_n} \rho_n\, dx + \int_{R_n \leqslant |x-y_n| \leqslant 4R_n} \zeta_n^2 \rho_n\, dx \\ &= \int \rho_n^{(1)}\, dx + O(\varepsilon) \\ &\geqslant \bar{\mu} + O(\varepsilon). \end{aligned}$$

In obtaining the third equality in the above derivation, we used Eq. (2.11) with η_n replaced by ζ_n, and the fact that the first two terms on the right-hand side of (2.11) are quantities of size $O(\varepsilon)$ as $R_n \to \infty$. Similarly,

$$\begin{aligned} E(u_n^{(2)}) &\geqslant \int [(u_n^{(2)})^2 + (M_2 u_n^{(2)})^2]\, dx \\ &= \int [(\phi_n u_n)^2 + (M_2(\phi_n u_n))^2]\, dx \\ &= O(\varepsilon) + \int \phi_n^2[u_n^2 + (M_2 u_n)^2]\, dx \\ &= O(\varepsilon) + \int \phi_n^2 \rho_n\, dx \\ &= O(\varepsilon) + \int_{|x-y_n| \geqslant 2R_n} \rho_n\, dx + \int_{R_n \leqslant |x-y_n| \leqslant 4R_n} \phi_n^2 \rho_n\, dx \end{aligned}$$

$$\begin{aligned}&= \int \rho_n^{(2)}\,dx + O(\varepsilon)\\&\geqslant \mu - \bar{\mu} + O(\varepsilon),\end{aligned}$$

which concludes our proof. $\square$

Lemma 2.16. *Assume that dichotomy holds for ρ_n. Then there exists $q_1 \in (0, q)$ such that*

$$I_q \geqslant I_{q_1} + I_{q-q_1}.$$

Proof. Let $\bar{q} = \bar{q}(\varepsilon)$ be the function defined in Lemma 2.15. Since $Q(u_n^{(1)})$ is bounded, the range of values of $\bar{q}(\varepsilon)$ remains bounded as $\varepsilon \to 0$. Therefore, by restricting ourselves to a sequence of values of ε tending to zero, and choosing an appropriate subsequence of this sequence, we may assume that $\bar{q}(\varepsilon)$ tends to a limit q_1 as $\varepsilon \to 0$.

We wish to show that $q_1 \in (0, q)$. To see this, we first observe that it follows from

$$E(u_n) = E(u_n^{(1)}) + E(u_n^{(2)}) + O(\varepsilon)$$

that

$$I_q = \liminf E(u_n) \geqslant \liminf E(u_n^{(1)}) + \liminf E(u_n^{(2)}) + O(\varepsilon). \tag{2.12}$$

Suppose now that $q_1 \leqslant 0$. Then for large n we have

$$Q(u_n^{(2)}) = q - q_1 + O(\varepsilon).$$

Let $\tilde{u}_n^{(2)} = \sigma_n u_n^{(2)}$, where σ_n is chosen so that $Q(\tilde{u}_n^{(2)}) = q - q_1$. Then $\sigma_n = 1 + O(\varepsilon)$ and

$$E(u_n^{(2)}) = \frac{1}{\sigma_n^2} E(\tilde{u}_n^{(2)}) \geqslant \frac{1}{\sigma_n^2} I_{q-q_1} \geqslant \frac{1}{(1 + O(\varepsilon))^2} I_q,$$

where the last inequality is due to Lemma 2.9. It follows from (2.12) that

$$I_q \geqslant \liminf E(u_n^{(1)}) + \frac{1}{(1 + O(\varepsilon))^2} I_q + O(\varepsilon).$$

But from (2.8a) we have

$$\liminf E(u_n^{(1)}) \geqslant \bar{\mu} + O(\varepsilon).$$

Hence we conclude that

$$I_q \geqslant \bar{\mu} + \frac{1}{(1 + O(\varepsilon))^2} I_q + O(\varepsilon),$$

and taking the limit as $\varepsilon \to 0$ gives

$$I_q \geqslant \bar{\mu} + I_q > I_q,$$

which is a contradiction.

On the other hand, if it were true that $q_1 \geqslant q$, then we would have $Q(u_n^{(1)}) = q_1 + O(\varepsilon)$ for large n, and an argument similar to that in the preceding paragraph would show that (2.12) implies

$$\begin{aligned} I_q &\geqslant \liminf E(u_n^{(2)}) + \frac{1}{(1+O(\varepsilon))^2} I_{q_1} + O(\varepsilon) \\ &\geqslant (\mu - \bar{\mu}) + \frac{1}{(1+O(\varepsilon))^2} I_q + O(\varepsilon), \end{aligned}$$

and hence

$$I_q \geqslant (\mu - \bar{\mu}) + I_q > I_q,$$

which is another contradiction. This completes the proof that $q_1 \in (0, q)$.

Finally, we see from the above arguments that

$$I_q \geqslant \frac{1}{(1+O(\varepsilon))^2} I_{q_1} + \frac{1}{(1+O(\varepsilon))^2} I_{q-q_1} + O(\varepsilon),$$

and taking the limit in the above equation as $\varepsilon \to 0$ gives

$$I_q \geqslant I_{q_1} + I_{q-q_1}$$

as desired. $\square$

Lemma 2.17. *Dichotomy does not occur.*

Proof. This follows immediately from Lemma 2.11 and Lemma 2.16. $\square$

We can now complete the proof of Theorem 2.2. Let $\{u_n\}$ be any minimizing sequence for I_q. Then by Lemmas 2.6, 2.13, and 2.17, we know that compactness occurs. That is, there exists a sequence of real numbers $\{y_n\}$ such that for any $\varepsilon > 0$, one can find $R > 0$ for which

$$\int_{|x-y_n| \leqslant R} \rho_n \, dx \geqslant \mu - \varepsilon \quad \text{for all } n,$$

or, in other words,

$$\int_{|x-y_n| \geqslant R} \rho_n \, dx \leqslant \varepsilon,$$

and hence

$$\int_{|x-y_n| \geqslant R} u_n^2 \, dx \leqslant \varepsilon.$$

Let $\tilde{u}_n(x) = u_n(x + y_n)$; then

$$\int_{|x|\geqslant R} \tilde{u}_n^2 \, dx \leqslant \varepsilon. \tag{2.13}$$

Since $\{\tilde{u}_n\}$ is a bounded sequence in $H^{\frac{s}{2}}(\mathbb{R})$, then by the Rellich lemma, on every bounded interval I there exists a subsequence of $\{\tilde{u}_n\}$ that converges to a function in $L^2(I)$. This fact, together with (2.13), enables us to carry out a Cantor diagonalization procedure to extract a subsequence of $\{\tilde{u}_n\}$ that converges to a function g in $L^2(\mathbb{R})$.

To see this, let $\varepsilon = \frac{1}{k}, k \in \mathbb{N}$. Then there exists $r_k > 0$ such that

$$\int_{[-r_k, r_k]^c} \tilde{u}_n^2 \, dx \leqslant \frac{1}{k}$$

for all n. By Rellich's lemma, for $k = 1$, there exist a function g_1 and a subsequence of $\{\tilde{u}_n\}$, denoted by $\{\tilde{u}_{1,n}\}$, such that $\tilde{u}_{1,n} \to g_1$ in $L^2[-r_1, r_1]$ and

$$\int_{[-r_1, r_1]^c} \tilde{u}_{1,n}^2 \, dx \leqslant 1 \quad \text{for all } n.$$

Inductively, for any $k \in \mathbb{N}$, there exist a function g_k and a subsequence of $\{\tilde{u}_{k-1,n}\}_{n\in\mathbb{N}}$, denoted by $\{\tilde{u}_{k,n}\}_{n\in\mathbb{N}}$, such that $\tilde{u}_{k,n} \to g_k$ in $L^2[-r_k, r_k]$ and

$$\int_{[-r_k, r_k]^c} \tilde{u}_{k,n}^2 \, dx \leqslant \frac{1}{k} \quad \text{for all } n.$$

Now for each $k \in \mathbb{N}$, choose n_k so that $\tilde{u}_{n_k}$ belongs to the subsequence $\{\tilde{u}_{k,n}\}_{n\in\mathbb{N}}$ and satisfies

$$||\tilde{u}_{n_k} - g_k||_{L^2[-r_i, r_i]} \leqslant \frac{1}{k}.$$

We claim that the sequence $\{\tilde{u}_{n_k}\}_{k\in\mathbb{N}}$ is Cauchy in $L^2(\mathbb{R})$. Indeed, for $k, l \geqslant K$, we have

$$\begin{aligned}
|\tilde{u}_{n_k} - \tilde{u}_{n_l}|_2^2 &= \int |\tilde{u}_{n_k} - \tilde{u}_{n_l}|^2 \, dx \\
&= \int_{[-r_K, r_K]} |\tilde{u}_{n_k} - \tilde{u}_{n_l}|^2 \, dx + \int_{[-r_K, r_K]^c} |\tilde{u}_{n_k} - u_{n_l}|^2 \, dx \\
&\leqslant 2 \int_{[-r_K, r_K]} |\tilde{u}_{n_k} - g_K|^2 \, dx + 2 \int_{[-r_K, r_K]} |\tilde{u}_{n_l} - g_K|^2 \, dx \\
&\quad + 2 \int_{[-r_K, r_K]^c} (\tilde{u}_{n_k})^2 \, dx + 2 \int_{[-r_K, r_K]^c} (\tilde{u}_{n_l})^2 \, dx \\
&\leqslant \frac{2}{K^2} + \frac{2}{K^2} + \frac{2}{K} + \frac{2}{K},
\end{aligned}$$

which proves the claim. Therefore $\{\tilde{u}_{n_k}\}$ converges in L^2 to some $g \in L^2(\mathbb{R})$.

We will now show that $g \in G_q$ and that a subsequence of $\{\tilde{u}_{n_k}\}$ converges to g in $H^{\frac{s}{2}}(\mathbb{R})$. To do this, first note that $\{\tilde{u}_{n_k}\}$ also converges to g in $L^{p+2}(\mathbb{R})$, since

$$\begin{aligned} |\tilde{u}_{n_k} - g|_{p+2} &\leqslant A\|\tilde{u}_{n_k} - g\|_{\frac{1}{2}} \\ &\leqslant A\|\tilde{u}_{n_k} - g\|_0^{1-\frac{1}{s}}\|\tilde{u}_{n_k} - g\|_{\frac{s}{2}}^{\frac{1}{s}} \\ &\leqslant A\|\tilde{u}_{n_k} - g\|_0^{1-\frac{1}{s}}, \end{aligned}$$

where we have used standard Sobolev imbedding and interpolation theorems and the fact that $\{\tilde{u}_{n_k}\}$ is bounded in $H^{\frac{s}{2}}(\mathbb{R})$. Since $Q(\tilde{u}_{n_k}) = q$ for all k, it follows that $Q(g) = q$. Also, since $\{\tilde{u}_{n_k}\}$ is bounded in $H^{\frac{s}{2}}(\mathbb{R})$, then some subsequence of $\{\tilde{u}_{n_k}\}$, which we also denote by $\{\tilde{u}_{n_k}\}$, converges to g weakly in $H^{\frac{s}{2}}(\mathbb{R})$. But from assumptions A2 and A3 on $m(k)$, it follows easily that the map $u \mapsto E(u)^{\frac{1}{2}}$ defines a norm on $H^{\frac{s}{2}}(\mathbb{R})$ which is equivalent to the standard norm. It then follows that

$$E(g)^{\frac{1}{2}} \leqslant \liminf E(\tilde{u}_{n_k})^{\frac{1}{2}},$$

so

$$E(g) \leqslant \liminf E(\tilde{u}_{n_k}) = I_q.$$

Hence $g \in G_q$, and

$$\lim_{n \to \infty} E(\tilde{u}_{n_k})^{\frac{1}{2}} = I_q^{\frac{1}{2}} = E(g)^{\frac{1}{2}}.$$

It now follows from the fact that $\{\tilde{u}_{n_k}\} \to g$ weakly in $H^{\frac{s}{2}}(\mathbb{R})$, the norm equivalency, and the preceding equality that $\{\tilde{u}_{n_k}\} \to g$ in $H^{\frac{s}{2}}(\mathbb{R})$.

3. Stability theory for general p

In this section, we prove the following theorem.

Theorem 3.1. *Suppose p is an arbitrary positive integer, and suppose the assumptions* A2–A4 *are satisfied by $m(k)$. Suppose also that $m(k)$ is a non-decreasing function of $|k|$. Then there exists $q_0 = q_0(p) \geqslant 0$ such that for all $q > q_0$, G_q is non-empty, and is stable in the sense of Corollary* 2.5. (q_0 *is defined in Lemma* 3.3.)

To prove Theorem 3.1, a new argument will be required to establish analogues of Lemmas 2.9–2.11. We have the following three lemmas.

Lemma 3.2. *For* $\theta \geqslant 1$ *and* $q > 0$, $I_{\theta q} \leqslant \theta I_q$ *and* $\bar{I}_{\theta q} \leqslant \theta \bar{I}_q$.

Proof. Let $\{\phi_n\}$ be a minimizing sequence for I_q and let $\psi_n(x) = \phi_n(\frac{x}{\theta})$. Then $Q(\psi_n(x)) = \theta q$ and a computation gives

$$E(\psi_n(x)) = \theta E(\phi_n(x)) - \theta \int \left[m(x) - m\left(\frac{x}{\theta}\right)\right] |\hat{\phi}_n(x)|^2 \, dx.$$

Since $m(k)$ is a non-decreasing function of $|k|$, we have

$$E(\psi_n(x)) \leqslant \theta E(\phi_n(x)),$$

and hence

$$I_{\theta q} \leqslant \theta I_q.$$

Now since $\bar{I}_q = I_q - 2q$, then $\bar{I}_{\theta q} \leqslant \theta \bar{I}_q$ for all $\theta \geqslant 1$ and $q > 0$. $\square$

Lemma 3.3. *For all* $q > 0$, $\bar{I}_q \leqslant 0$. *Moreover, either there exists a* $q_0 > 0$ *such that* $\bar{I}_q = 0$ *for* $0 < q < q_0$ *and* $\bar{I}_q < 0$ *for all* $q > q_0$, *or* $\bar{I}_q < 0$ *for all* $q > 0$ (*in which case we define* $q_0 = 0$). *In either case, the conclusion of Lemma* 2.10 *holds for any minimizing sequence of* I_q, *provided* $q > q_0$.

Proof. First we show that $\bar{I}_q \leqslant 0$ for all $q > 0$. To see this, let ϕ be a positive function in $H^{\frac{s}{2}}(\mathbb{R})$ such that $\frac{1}{2}\int \phi^2 \, dx = q$. Define $\phi_n(x) = \frac{1}{\sqrt{n}}\phi(\frac{x}{n})$. Then

$$Q(\phi_n) = \frac{1}{2}\int \phi^2 \, dx + \frac{1}{(p+1)(p+2)n^{\frac{p}{2}}} \int \phi^{p+2} \, dx,$$

so $Q(\phi_n) \to q$ as $n \to \infty$. But

$$\bar{E}(\phi_n) = \int m(k/n)|\hat{\phi}(k)|^2 \, dk - \frac{1}{(p+1)(p+2)n^{\frac{p}{2}}} \int \phi^{p+2} \, dx,$$

and the first integral on the right-hand side tends to zero as $n \to \infty$ by the Dominated Convergence Theorem, while the second integral also tends to zero as $n \to \infty$. Therefore $\lim_{n\to\infty} \bar{E}(\phi_n) = 0$, which shows that $\bar{I}_q \leqslant 0$.

Now let $S = \{q > 0 | \bar{I}_q = 0\}$. If S is empty, then $\bar{I}_q < 0$ for all $q > 0$, so we may assume S is non-empty. We claim that S is bounded above. To see this, fix a positive function ϕ in $H^{\frac{s}{2}}(\mathbb{R})$. For $q > 0$, choose $a = a(q) > 0$ such that $Q(a\phi) = q$. Note that $a(q) \to \infty$ as $q \to \infty$. Now

$$\bar{E}(a\phi) = a^2 \int \phi M \phi \, dx - \frac{2a^{p+2}}{(p+1)(p+2)} \int \phi^{p+2} \, dx,$$

and hence $\bar{E}(a\phi) < 0$ when q is sufficiently large. This shows that $\bar{I}_q < 0$ for large q, as desired.

Now let $q_0 = \sup S$. Then it is easy to see from Lemma 3.2 that $\bar{I}_q = 0$ for $0 < q < q_0$ and $\bar{I}_q < 0$ for $q > q_0$.

Finally, if $q > q_0$, then in either case $\bar{I}_q < 0$, from which the conclusion of Lemma 2.10 follows as shown in the proof of Lemma 2.10. □

Lemma 3.4. *If $q > q_0$, $q_1 > 0, q_2 > 0$ and $q_1 + q_2 = q$, then $I_q < I_{q_1} + I_{q_2}$.*

Proof. We shall show $\bar{I}_q < \bar{I}_{q_1} + \bar{I}_{q_2}$, from which the Lemma follows immediately, since $\bar{I}_q = I_q - 2q$.

We may assume that one of $\bar{I}_{q_1}$ and $\bar{I}_{q_2}$ is less than 0, say $\bar{I}_{q_1}$. (Otherwise, since $\bar{I}_q < 0$, then $\bar{I}_q < \bar{I}_{q_1} + \bar{I}_{q_2}$ is obvious.) Then the conclusion of Lemma 2.10 holds for any minimizing sequence of $\bar{I}_{q_1}$. Therefore we can use the same argument as in the first part of the proof of Lemma 2.11 to show that $I_{\theta q_1} < \theta I_{q_1}$ for all $\theta > 1$. Hence if $q_1 \geqslant q_2$, then

$$\begin{aligned}
\bar{I}_q &= \bar{I}_{q_1+q_2} \\
&= \bar{I}_{q_1\left(1+\frac{q_2}{q_1}\right)} \\
&< \left(1 + \frac{q_2}{q_1}\right)\bar{I}_{q_1} \\
&= \bar{I}_{q_1} + \frac{q_2}{q_1}\bar{I}_{q_1} \\
&\leqslant \bar{I}_{q_1} + \frac{q_2}{q_1}\frac{q_1}{q_2}\bar{I}_{q_2} \\
&= \bar{I}_{q_1} + \bar{I}_{q_2}.
\end{aligned}$$

If $q_1 < q_2$, then by Lemma 3.2, $\bar{I}_{q_2} \leqslant \frac{q_2}{q_1}\bar{I}_{q_1} < 0$ and we can just interchange q_1 and q_2 in the above argument. □

With Lemma 3.4 in hand, we can now complete the proof of Theorem 3.1 by following the proof of Theorem 2.2 and its corollaries. Lemmas 2.7 and 2.8 remain valid in our present situation, and in place of Lemma 2.11 we have Lemma 3.4. We can now rule out vanishing using Lemmas 2.12 and 2.13 as before (the proof of Lemma 2.13 is still valid, because when $q > q_0$ we can substitute Lemma 3.3 for Lemma 2.10). To rule out dichotomy, we note that Lemmas 2.14–2.16 still hold; and Lemmas 3.14 and 2.16 show that dichotomy leads to a contradiction, provided $q > q_0$. The proof then concludes as before.

As an illustration of the application of Theorem 3.1, as well as some of its limitations, we will in the remainder of this section consider the example of the generalized BBM equation (1.5), repeated here for convenience:

$$u_t + u_x + u^p u_x - u_{xxt} = 0.$$

The functionals of the variational problem associated with this equation are

$$E(u) = \int [u^2 + (u_x)^2]\, dx$$

and

$$Q(u) = \int \left[\frac{u^2}{2} + \frac{u^{p+2}}{(p+1)(p+2)}\right] dx,$$

and the functional $\bar{E}$ now becomes

$$\bar{E}(u) = \int \left[(u_x)^2 - \frac{2u^{p+2}}{(p+1)(p+2)}\right] dx.$$

For the generalized BBM equation, if the set of minimizers G_q is non-empty in $H^1(\mathbb{R})$ for some $q > 0$, then for any $g \in G_q$, we have (see Corollary 2.4)

$$-cg'' + (c-1)g - \frac{g^{p+1}}{p+1} = 0, \tag{3.1}$$

where c is used for $\frac{2}{\lambda}$. That is, g is a solitary-wave profile function with wavespeed c and will be rewritten as ϕ_c in what follows.

For each $c > 1$, Eq. (3.1) has a solution which is unique up to a translation and is given by

$$\phi_c = \sigma \operatorname{sech}^{\frac{2}{p}}(\tau x),$$

where

$$\sigma = \left[\frac{c-1}{2}(p+1)(p+2)\right]^{\frac{1}{p}}$$

and

$$\tau = \frac{p}{2}\sqrt{\frac{c-1}{c}}.$$

For ease of notation in what follows, we denote $Q(\phi_c)$ by $Q(c)$ and $\bar{E}(\phi_c)$ by $\bar{E}(c)$. A computation gives

$$\begin{aligned} Q(c) =& \frac{1}{p}\left[\frac{(p+1)(p+2)}{2}\right]^{\frac{2}{p}} \\ &\times \left[\sqrt{c}(c-1)^{\frac{4-p}{2p}} I\left(\frac{4}{p}\right) + \sqrt{c}(c-1)^{\frac{p+4}{2p}} I\left(\frac{2p+4}{p}\right)\right] \end{aligned}$$

and

$$\bar{E}(c) = \frac{4\sigma^2\tau}{p^2}k(p)I\left(\frac{4}{p}\right) - \frac{2\sigma^{p+2}}{(p+1)(p+2)\tau}I\left(\frac{2p+4}{p}\right),$$

where

$$I(p) = \int \operatorname{sech}^p(x)\,dx$$

and

$$k(p) = \frac{\int \operatorname{sech}^{\frac{4}{p}}x\tanh^2 x\,dx}{\int \operatorname{sech}^{\frac{4}{p}}x\,dx}.$$

From [20,15], we see that

$$I(p) = \frac{\Gamma(\frac{1}{2})\Gamma(\frac{r}{2})}{\Gamma(\frac{r+1}{2})}$$

and

$$k(p) = \frac{p}{4+p}.$$

With the help of the above identities, one can show that $\bar{E}(c)<0$ when $c>\frac{p}{4}$, $\bar{E}(c)=0$ when $c=\frac{p}{4}$, and $\bar{E}(c)>0$ when $c=\frac{p}{4}$.

Suppose $p<4$. By Theorem 2.2, G_q exists for all $q>0$. In this case $Q(c)$ is an increasing function on $(1,+\infty]$, $\lim_{c\to 1}Q(c)=0$ and $\lim_{c\to\infty}Q(c)=+\infty$. So for any $q>0$, G_q consists of only translates of ϕ_c with the unique speed c determined by $Q(c)=q$. It then follows that the solitary waves are individually stable for all $c>1$.

Next, suppose $p=4$. Then $Q(c)$ is again an increasing function on $(1,+\infty)$, with

$$q_0 = \lim_{c\to 1} Q(c) = \frac{1}{p}\left[\frac{(p+1)(p+2)}{2}\right]^{\frac{2}{p}} I\left(\frac{4}{p}\right),$$

and $\lim_{c\to\infty}Q(c)=+\infty$. Since $Q(c)>q_0$ for all $c>1$, then G_q is empty for all $q\leqslant q_0$. For $q>q_0$, there exists a unique solitary wave profile with speed $c>\frac{p}{4}=1$ such that $Q(c)=q$. It then follows from $\bar{E}(c)<0$ that $\bar{I}_q<0$. Hence, by Theorem 3.1, G_q is non-empty for all $q>q_0$ and solitary waves are individually stable for all $c>1$.

Finally, suppose $p>4$. Differentiating $Q(c)$ with respect to c gives

$$Q'(c) = 2(p+4)\left[\frac{(p+1)(p+2)}{2}\right]^{\frac{2}{p}} I\left(\frac{4}{p}\right)\frac{(c-1)^{\frac{4-3p}{2p}}}{\sqrt{c}} \times [(8p+16)c^2 - 8pc - p^2].$$

The only solution to $Q'(c) = 0$ that is greater that 1 is

$$c_r = \frac{p}{2p+4}\left(1 + \sqrt{\frac{4+p}{2}}\right),$$

and we see that $Q'(c) > 0$ when $c > c_r$ and $Q'(c) < 0$ when $1 < c < c_r$. So $Q(c)$ decreases on $(1, c_r)$, achieves minimum value at c_r and increases on $(c_r, +\infty)$; and for any $q > q_r = Q(c_r)$, there are two numbers c_1 and c_2 such that $1 < c_1 < c_r < c_2$ and $Q(c_1) = Q(c_2) = q$. Also note that when $p > 4$

$$c_0 = \frac{p}{4} > c_r = \frac{p}{2p+4}\left(1 + \sqrt{\frac{4+p}{2}}\right).$$

It is now clear that G_q does not exist for all $q < q_r$. If $q > q_0 = Q(c_0)$, $\bar{I}_q < 0$; so G_q exists. Of the two solitary wave speeds c_1 and c_2 with the property that $1 < c_1 < c_r < c_2$ and $Q(c_1) = Q(c_2) = q > q_0$, only c_2 satisfies $\bar{E}(c_2) < 0$. Therefore, G_q consists of only translates of the solitary wave profile with wavespeed c_2 and it follows that all solitary waves with wavespeed greater then c_0 are individually stable. Our next claim is that $\bar{I}_q = 0$ for $0 < q \leqslant q_0$. For if not, by Lemma 3.3, $\bar{I}_q$ would be negative and G_q would be non-empty and contain translates of a solitary wave profile of wavespeed c satisfying $\bar{E}(c) \geqslant 0$ ($\bar{E}(c) \geqslant 0$ for $q < q_0$, $\bar{E}(c) \geqslant 0$ for $q = q_0$), which is a contradiction. It then follows that G_q is empty for $q < q_0$ and non-empty for $q = q_0$. Thus we can extend our stability result by including c_0 into the range of wavespeeds of stable solitary waves. (Again, there are only translates of the wave profile of speed c_0 in G_{q_0}, since if c is the other wavespeed with $Q(c) = q_0$, then $\bar{E}(c) > 0$.)

As mentioned in the introduction of this paper, in the case that $p > 4$, it was proved by Souganidis and Strauss that the solitary waves are stable for all $c > c_r$ and unstable for $c \leqslant c_r$. We have, using a different approach, recovered the stability result for wavespeeds greater than or equal to c_0. Since G_q does not exist for $q < q_0$, our method is not able to show the stability of solitary waves for wavespeeds greater than c_r and less than c_0. On the other hand, we have completely solved the variational problem associated with the generalized BBM equation for all positive integer values of p and all $q > 0$. It is also interesting to observe that while solitary waves with wavespeed greater then c_r are stable, the profile functions of those with speed greater than or equal to c_0 are minimizers of the variational problem and the profiles of those with speed less than c_0 are not.

4. Further results

If $f(u) = \frac{u^{p+1}}{p+1}$, Eq. (1.1) reduces to

$$u_t + u^p u_x + M u_t = 0. \tag{4.1}$$

We are to minimize

$$E(u) = \int (u^2 + uMu)\,dx,$$

where $u \in H^{\frac{s}{2}}(\mathbb{R})$ is subjected to the constraint

$$Q(u) = \int \tfrac{u^{p+2}}{(p+1)(p+2)}\,dx = q.$$

As before, we let G_q stand for the set of minimizers of $E(u)$ subject to this constraint.

Theorem 4.1. *Suppose* $f(u) = \frac{u^{p+1}}{p+1}$, *and* $m(k)$ *satisfies assumptions* A2–A4. *Then for every* $q > 0$, G_q *is non-empty, and is a stable set of solitary-wave solutions of* (4.1), *in the sense of Corollary* 2.5. *If p is odd, then the result also holds for all* $q<0$.

Proof. If $q > 0$, then all the lemmas that have been proved in Section 2 are still valid. The first condition was used to prove Lemma 2.10 and is no longer necessary, since the lemma is obviously true. It is straightforward to modify the proofs of Lemmas 2.7, 2.11, and 2.15. The proofs for the other lemmas remain unchanged. The theorem then follows in the same way as Theorem 2.2 and its corollaries.

For p odd and $q<0$, one simply notes that $\{u_n\}$ is a minimizing sequence for I_q if and only if $\{-u_n\}$ is a minimizing sequence for I_{-q}; the result then follows from the result for $q > 0$. $\square$

The proofs of the stability results we have stated so far have relied on assumption A3, the non-negativity of the symbol $m(k)$. We now give an example showing how the theory may be adapted to a situation in which A3 does not apply. Consider the equation

$$u_t + u_x + uu_x + Mu_t = 0, \tag{4.2}$$

where the symbol $m(k)$ is given by $m(k) = k^2 - \alpha|k|$ for $\alpha<2$. The variational problem for this equation is to minimize

$$E(u) = \int (u^2 + uMu)\,dx$$

over the set all $u \in H^1(\mathbb{R})$ satisfying the constraint

$$Q(u) = \int \left(\frac{u^2}{2} + \frac{u^3}{6}\right) dx = q.$$

As before, we let G_q denote the set of minimizers, if any exist.

Theorem 4.2. *For every $q > 0$, G_q is non-empty, and is a stable set of solitary-wave solutions of* (4.2).

Proof. For this $m(k)$, there again exists $C_1 > 0$ and $C_2 > 0$ such that

$$C_1(1 + k^2) \leqslant 1 + m(k) \leqslant C_2(1 + k^2)$$

for all $k \in \mathbb{R}$. So Lemma 2.7, with a replacement of $H^{\frac{s}{2}}(\mathbb{R})$ norm by $H^1(\mathbb{R})$ norm, and Lemma 2.8 are still true.

Let

$$\begin{aligned}\bar{E}(u) &= E(u) - 2\left(1 - \frac{\alpha^2}{4}\right)Q(u)\\ &= \int \left[\left(k - \frac{\alpha}{2}\right)^2 |\hat{u}(k)|^2 - \frac{4 - \alpha^2}{12} u^3(k)\right] dk\end{aligned}$$

and

$$\bar{I}_q = \inf\{E(u) | u \in H^1(\mathbb{R}) \text{ and } Q(u) = q\}.$$

We claim that $\bar{I}_q < 0$. To prove this, it suffices to find a function ϕ such that $Q(\phi) = q$ and $\bar{E}(\phi) < 0$. We construct such a ϕ following the lines of similar constructions in [3] and [5]. To begin with, we consider the case when q is small. Let

$$\phi = \phi_\varepsilon = ah(\varepsilon x)(\cos k_0 x + \varepsilon),$$

where

$$h(x) = \frac{1}{1 + x^2},$$

$\varepsilon = q^2$, $k_0 = \frac{\alpha}{2}$, and a is chosen such that $Q(\phi) = q$. By considering the behavior of both sides of the equation of $Q(\phi) = q$ for small ε, one finds that $a \sim \varepsilon^{\frac{3}{4}}$ as $\varepsilon \to 0$ (cf. the proof of Eq. (3) in [3]; the presence of the cubic term in our expression for $Q(u)$ does not affect this estimate). Therefore, the general computation made in the proof of Theorem 2 of [3] applies without modification in this case, and shows that

$$\int \left(k - \frac{\alpha}{2}\right)^2 |\hat{\phi}(k)|^2\, dk = O(\varepsilon^{\frac{5}{2}}),$$

while

$$\int \phi^3\, dx \geqslant A\varepsilon^{\frac{9}{4}}$$

for sufficiently small ε, where $A > 0$. Hence there exists ε_0 such that $\bar{E}(\phi_\varepsilon) < 0$ for $\varepsilon \in (0, \varepsilon_0]$. This proves that $\bar{I}_q < 0$ for $q \in (0, q_0]$, where $q_0 = \varepsilon_0^2$. Now let $\phi = \phi_{\varepsilon_0}$, and let q be given such that $q \geqslant q_0$. Since $\int \phi^3\, dx > 0$, and $Q(\phi) = \int (\frac{\phi^2}{2} + \frac{\phi^3}{6})\, dx = q_0$, we can find $\beta \geqslant 1$ such that $Q(\beta\phi) = q$. Then

$$\begin{aligned}
\bar{E}(\beta\phi) &= \int \left[\beta^2 \left(k - \frac{\alpha}{2}\right)^2 |\hat{\phi}(k)|^2 - \beta^3 \left(\frac{4-\alpha^2}{12}\right) \phi^3(k)\right] dk \\
&\leqslant \beta^2 \int \left[\left(k - \frac{\alpha}{2}\right)^2 \left[|\hat{\phi}(k)|^2 - \left(\frac{4-\alpha^2}{12}\right) \phi^3(k)\right] dk \\
&= \beta^2 \bar{E}(\phi) \\
&< 0,
\end{aligned}$$

and it follows that $\bar{I}_q < 0$.

Now let $\{u_n\}$ be any minimizing sequence for the constrained variational problem, and define $\rho_n = u_n^2 + (u_n)_x^2$ so that $\int \rho_n\, dx = ||u_n||_1^2$. By passing to a subsequence, we may assume there exists $\mu > 0$ such that $\int \rho_n\, dx \to \mu$. The proof of Lemma 2.9 goes through as before, and since we have established above that $\bar{I}_q < 0$ for all $q > 0$, then the argument in the last paragraph of the proof of Lemma 2.10 shows that the statement of Lemma 2.10 still holds. This is enough for the proof of Lemma 2.13 to be carried out as before, so we have shown that vanishing does not occur for $\{u_n\}$.

To complete the proof of Theorem 4.2, then, it remains only to show that dichotomy cannot occur for $\{u_n\}$. For this, we first note that Lemma 2.11 and its proof remain valid. Next. instead of using the proof of Lemma 2.15, we can use the argument given in the proof of Theorem 2.5 of [10] to show that the conclusion of Lemma 2.15 still holds in the present situation. Finally, the proof of Lemma 2.16 proceeds as above in Section 2, and thus dichotomy is ruled out. $\square$

Acknowledgments

I would like to deeply thank John P. Albert for his guidance and encouragement. He contributed tremendously to this paper and kindly insisted not to be included as a co-author. I also would like to thank the referee for his valuable comments.

References

[1] J. Albert, Concentration compactness and the stability of solitary-wave solutions to nonlocal equations, Applied Analysis (Baton Rouge, LA, 1996) Contemporary Mathematics, Vol. 221, American Mathematical Society, Providence, RI, 1999, pp. 1–29.

[2] J.P. Albert, J.L. Bona, Comparisons between model equations for long waves, J. Nonlinear Sci. 1 (1991) 345–374.

[3] J.P. Albert, F. Linares, Stability of solitary-wave solutions of long-wave equations with general dispersion, Mat. Contemp. 15 (1998) 1–19.

[4] J.P. Albert, F. Linares, Stability and symmetry of solitary-wave solutions to systems modeling interactions of long waves, J. Math. Pures Appl. 79 (2000) 195–226.

[5] J. Angulo Pava, Existence and stability of solitary wave solutions of the Benjamin equation, J. Differential Equations 152 (1999) 136–159.
[6] T.B. Benjamin, The stability of solitary waves, Proc. Roy. Soc. London Ser. A 328 (1972) 153–183.
[7] T.B. Benjamin, J.L. Bona, J.J. Mahony, Model equations for long waves in nonlinear dispersive systems, Philos. Trans. Roy. Soc. London Ser. A 272 (1972) 47–78.
[8] J. Bona, On the stability theory of solitary waves, Proc. Roy. Soc. London Ser. A 344 (1975) 363–374.
[9] J.L. Bona, P.E. Souganidis, W.A. Strauss, Stability and instability of solitary waves of KdV type, Proc. Roy. Soc. London A 411 (1987) 395–412.
[10] T. Cazenave, P.-L. Lions, Orbital stability of standing waves for some nonlinear Schrödinger equations, Comm. Math. Phys. 85 (1982) 549–561.
[11] H. Chen, J.L. Bona, Existence and asymptotic properties of solitary-wave solutions of Benjamin-type equations, Adv. Differential Equations 3 (1998) 51–84.
[12] R. Coifman, Y. Meyer, Au delà des opérateurs pseudo-différentiels Asterisque No. 57, Société Mathématique de France, Paris, 1978.
[13] A. de Bouard, J.-C. Saut, Remarks on the stability of generalized KP solitary waves, Contemp. Math. 200 (1996) 75–84.
[14] A. de Bouard, J.-C. Saut, Solitary waves of generalized Kadomtsev–Petviashvili equations, Ann. Inst. H. Poincaré Anal. Non Linéaire 14 (1997) 211–236.
[15] I.S. Gradshetyn, I.M. Ryzhik, Tables of Integrals, Series and Products, Academic Press, New York, 1980.
[16] D.J. Korteweg, G. de Vries, On the change of form of long waves advancing in a rectangular canal, and on a new type of long stationary waves, Philos. Mag. 39 (1895) 422–443.
[17] P. Lions, The concentration compactness principle in the calculus of variations, The locally compact case, part 1, Ann. Inst. H. Poincaré 1 (1984) 109–145.
[18] P. Lions, The concentration compactness principle in the calculus of variations, The locally compact case, part 2, Ann. Inst. H. Poincaré 4 (1984) 223–283.
[19] D. Luenberger, Optimization by Vector Space Methods, Wiley, New York, 1969.
[20] R.L. Pego, M.I. Weinstein, Eigenvalues, and instabilities of solitary waves, Philos. Trans. Roy. Soc. London A 340 (1992) 47–94.
[21] P.E. Souganidis, W.A. Strauss, Instability of a class of dispersive solitary waves, Proc. Roy. Soc. Edinburgh A 114 (1990) 195–212.
[22] M.I. Weinstein, Lyapunov stability of ground states of nonlinear dispersive evolution equations, Comm. Pure Appl. Math. 39 (1986) 51–68.
[23] M.I. Weinstein, On the structure and formation of singularities in solutions to nonlinear dispersive evolution equations, Comm. Partial Differential Equations 11 (1986) 545–565.